

# Hypergeometric Solutions for the $q$ -Painlevé Equation of type $E_6^{(1)}$ by Padé Method

Yusuke Ikawa

Department of Mathematics, Faculty of Science, Kobe University, Hyogo 657-8501, Japan

Email: yikawa@math.kobe-u.ac.jp

**Abstract:** The  $q$ -Painlevé equation of type  $E_6^{(1)}$  is obtained by Padé method. Special solutions in determinant formula to the  $q$ -Painlevé equation is presented. A relation between Padé method and Bäcklund transformation of type  $E_6^{(1)}$  is given.

**Mathematics Subject Classifications (2010):** 34A05, 41A21, 34M55

**Key words:**  $q$ -Painlevé equations, Padé method, QRT system, Bäcklund transformation

## 1 Introduction

$q$ -Painlevé equations and their solutions have been studied from various viewpoints. Especially, Sakai gave a natural classification of discrete Painlevé equations by means of the geometry of rational surfaces [14]. In this classification, discrete Painlevé equations are classified by means of affine Weyl groups.

For the solutions of  $q$ -Painlevé equations, seed solutions of hypergeometric type were obtained (see for instance [6]). More general solutions obtained by applying the Bäcklund transformations have been given in [1][2][4][5][7][8][9][13]. These results are based on the bilinear relations for the  $\tau$ -functions.

$q$ -Painlevé equation of type  $E_6^{(1)}$ , a main subject in this paper is first proposed in [11]. We study the equation in the form appeared in [12] i.e. the equations (84) and (85) where  $f_1$  and  $g_1$  are variables and  $b_1, \dots, b_8$  are parameters. And in [6], a hypergeometric solution to the  $q$ -Painlevé equations is expressed in terms of the  $q$ -hypergeometric series (46). Though the seed solution (46) for the  $q$ -Painlevé equation is already known, any results have not been obtained for its Bäcklund transformed solutions so far.

In this paper, we will construct the hypergeometric solutions to the  $q$ -Painlevé equation of type  $E_6^{(1)}$  in a determinant formula by using Padé method [15][16], which is one method to derive Lax equations to the Painlevé equation. Applying this method to  $q$ -Painlevé equations were not accomplished for  $E_6^{(1)}$  case before.

This paper is organized as follows. In section 2, we will get  $q$ -Painlevé VI equation[3] by Padé method (Theorem2.2) as warming up example. Using this method, we will obtain  $q$ -Painlevé equation of type  $E_6^{(1)}$  (Theorem3.2) and its explicit solutions in determinant formula (Theorem 3.6) in section 3. In appendix A, we will discuss the relation between QRT

system[10] and  $q$ -Painlevé equations. Finally in appendix B, we will study a relation between the Padé method and the Bäcklund transformation.

## 2 The case of $D_5^{(1)}$ ( $q$ -Painlevé VI)

In this section, we will derive the Lax pair for  $D_5^{(1)}$   $q$ -Painlevé equation (Theorem 2.1) and the  $q$ -Painlevé VI equation (Theorem 2.2) by using the Padé method.

### 2.1 Construction of the difference equation

Let  $a_1, a_2, a_3, a_4 \in \mathbb{C} \setminus \{0\}$  be complex parameters. Define a function  $Y(x)$  as

$$Y(x) = \frac{(a_1x, a_2x)_\infty}{(a_3x, a_4x)_\infty} \quad (1)$$

where the symbol  $(\ )_i$  is defined as

$$(z_1, z_2, \dots, z_i)_j = \prod_{k=0}^j (1 - q^k z_1)(1 - q^k z_2) \cdots (1 - q^k z_i).$$

Let  $m, n \in \mathbb{Z}_{\geq 0}$  and put  $N = m + n$ . We consider the approximation of  $Y(x)$  by rational function

$$Y(x) \equiv \frac{P_m(x)}{Q_n(x)} \pmod{x^{N+1}} \quad (2)$$

where  $P_m(x)$  and  $Q_n(x)$  are polynomials of order  $m$  and  $n$  respectively and the constant term of  $Q_n(x)$  is 1.

We will construct two difference equations whose solutions are  $P_m(x)$  and  $Y(x)Q_n(x)$ .

We define a transformation  $T$  as

$$T : a_2 \mapsto qa_2, \quad a_4 \mapsto qa_4$$

and we denote as  $T(y) = \bar{y}$  and  $T^{-1}(y) = \underline{y}$ . Here and in the following, we will write only non-trivial actions. Using this transformation  $T$ , we will construct following difference equations

$$L_2 : \begin{vmatrix} y(x) & y(qx) & \bar{y}(x) \\ P_m(x) & P_m(qx) & \bar{P}_m(x) \\ Y(x)Q_n(x) & Y(qx)Q_n(qx) & \bar{Y}(x)\bar{Q}_n(x) \end{vmatrix} = 0, \quad (3)$$

$$L_3 : \begin{vmatrix} y(x) & \bar{y}(x) & \bar{y}(x/q) \\ P_m(x) & \bar{P}_m(x) & \bar{P}_m(x/q) \\ Y(x)Q_n(x) & \bar{Y}(x)\bar{Q}_n(x) & \bar{Y}(x/q)\bar{Q}_n(x/q) \end{vmatrix} = 0. \quad (4)$$

**Theorem 2.1** *The explicit form of the equations  $L_2$  and  $L_3$  are obtained as*

$$L_2 : g(a_4x)_1 y(x) - (a_1x)_1 y(qx) + c_1(xf)_1 \bar{y}(x) = 0, \quad (5)$$

$$L_3 : c_2 \left( \frac{x\bar{f}}{q} \right)_1 y(x) + g(a_2x)_1 \bar{y}(x) - q^{N+1} \left( \frac{a_3}{q} \right)_1 \bar{y} \left( \frac{x}{q} \right) = 0 \quad (6)$$

where  $f, g, c_1$  and  $c_2$  are constant with respect to  $x$  and  $\bar{f} = T(f)$ .

**Proof**

First, we will consider the equation  $L_2$ . By definition of  $Y(x)$ , we have

$$\frac{Y(qx)}{Y(x)} = \frac{(a_3x, a_4x)_1}{(a_1x, a_2x)_1}, \quad \frac{\overline{Y}(x)}{Y(x)} = \frac{(a_4x)_1}{(a_2x)_1}.$$

Then the coefficient of  $\overline{y}(x)$  in  $L_2$  is computed as follows,

$$\begin{aligned} & Y(qx)P_m(x)Q_n(qx) - Y(x)P_m(qx)Q_n(x) \\ &= Y(x) \left\{ \frac{Y(qx)}{Y(x)} P_m(x)Q_n(qx) - P_m(qx)Q_n(x) \right\} \\ &= Y(x) \left\{ \frac{(a_3x, a_4x)_1}{(a_1x, a_2x)_1} P_m(x)Q_n(qx) - P_m(qx)Q_n(x) \right\} \\ &= \frac{Y(x)}{(a_1x, a_2x)_1} \left\{ (a_3x, a_4x)_1 P_m(x)Q_n(qx) - (a_1x, a_2x)_1 P_m(qx)Q_n(x) \right\}. \end{aligned}$$

The part in  $\{ \}$  of the last expression is a polynomial in  $x$  of order  $N+2$ . By definition for Padé approximation, this polynomial has zero at  $x=0$  of order  $N+1$ . So, the coefficient of  $\overline{y}(x)$  is given by

$$A(x) := Y(qx)P_m(x)Q_n(qx) - Y(x)P_m(qx)Q_n(x) = \frac{c_0 x^{N+1} Y(x)}{(a_1x, a_2x)_1} (xf)_1 \quad (7)$$

where  $c_0$  and  $f$  are some constant.

Similarly, the coefficients of  $y(qx)$  and  $y(x)$  are given by

$$B(x) := \overline{Y}(x)P_m(x)\overline{Q}_n(x) - Y(x)\overline{P}_m(x)Q_n(x) = \frac{cY(x)x^{N+1}}{(a_2x)_1}, \quad (8)$$

$$C(x) := \overline{Y}(x)P_m(qx)Q_n(x) - Y(qx)\overline{P}_m(x)Q_n(qx) = \frac{c'Y(x)(a_4x)_1 x^{N+1}}{(a_1x, a_2x)_1} \quad (9)$$

where  $c, c'$  are some constant. By putting  $g = c'/c$ ,  $c_1 = c_0/c$ , we obtain  $L_2$  in the equation (5).

Next, we will consider the equation  $L_3$ . By definition.

$$\frac{Y(x/q)}{Y(x)} = \frac{\left(\frac{a_1}{q}x, \frac{a_2}{q}x\right)_1}{\left(\frac{a_3}{q}x, \frac{a_4}{q}x\right)_1}, \quad \frac{\overline{Y}(x/q)}{Y(x)} = \frac{\left(\frac{a_1}{q}x\right)_1}{\left(\frac{a_3}{q}x\right)_1}.$$

The coefficient of  $y(x)$  is

$$\begin{aligned} & \overline{Y}(x/q)\overline{P}_m(x)\overline{Q}_n(x/q) - \overline{Y}(x)\overline{P}_m(x/q)\overline{Q}_n(x) = -\overline{A}\left(\frac{x}{q}\right) \\ &= -\frac{\overline{c_0}Y(x)}{\left(\frac{a_3}{q}x, a_2x\right)_1} \left(\frac{x}{q}\right)^{N+1} \left(\frac{xf}{q}\right)_1. \end{aligned}$$

The coefficient of  $\bar{y}(x)$  is

$$\begin{aligned}\bar{Y}(x/q)P_m(x)Q/n(x/q) - Y(x)\bar{P}_m(x/q)Q_n(x) &= C\left(\frac{x}{q}\right) \\ &= \frac{c'Y(x)}{\left(\frac{a_3}{q}x\right)_1} \left(\frac{x}{q}\right)^{N+1}.\end{aligned}$$

The coefficient of  $\bar{y}(x/q)$  in  $L_3$  is  $B(x)$ . So, we get  $L_3$  when we simplify coefficients where  $c_2 = \bar{c}_0/c$ .  $\square$

## 2.2 Compatibility condition

We will consider the compatibility condition of the equations (5) and (6).

**Theorem 2.2** *The compatibility condition of the equations (5) and (6) is given by*

$$g\bar{g} = \frac{q^{N+1}\left(\frac{a_1}{\bar{f}}, \frac{a_3}{\bar{f}}\right)_1}{\left(\frac{qa_2}{\bar{f}}, \frac{qa_4}{\bar{f}}\right)_1}, \quad (10)$$

$$f\bar{f} = \frac{a_1a_3\left(\frac{a_4}{a_1q^m}g, \frac{a_2}{a_3q^n}g\right)_1}{\left(g, \frac{g}{q^{N+1}}\right)_1}. \quad (11)$$

*This is a  $q$ -Painlevé VI equation [3].*

### Proof

First, we will prove the equation (10). Translate  $L_2$  by  $T$ , we have

$$\bar{L}_2 : \bar{g}(qa_4x)_1\bar{y}(x) - (a_1x)_1\bar{y}(qx) + \bar{c}_1(x\bar{f})_1\bar{\bar{y}}(x) = 0.$$

We put  $x = 1/\bar{f}$  in this equation, then we have

$$\bar{g}\left(\frac{qa_4}{\bar{f}}\right)_1\bar{y}\left(\frac{1}{\bar{f}}\right) - \left(\frac{a_1}{\bar{f}}\right)_1\bar{y}\left(\frac{q}{\bar{f}}\right) = 0. \quad (12)$$

On the other hand, putting  $x = q/\bar{f}$  in  $L_3$ , we have

$$g\left(\frac{qa_2}{\bar{f}}\right)_1\bar{y}\left(\frac{q}{\bar{f}}\right) - q^{N+1}\left(\frac{a_3}{\bar{f}}\right)_1\bar{y}\left(\frac{1}{\bar{f}}\right) = 0. \quad (13)$$

Hence,

$$g\bar{g} = \frac{q^{N+1}\left(\frac{a_1}{\bar{f}}, \frac{a_3}{\bar{f}}\right)_1}{\left(\frac{qa_2}{\bar{f}}, \frac{qa_4}{\bar{f}}\right)_1}. \quad (14)$$

Then, we will prove the equation (11).  $P_m(x)$  is a solution of  $L_2$  and  $L_3$ , so we substitute for  $P(x) = k_0 + \cdots + kx^m$  in the  $L_2$  and  $L_3$  and check the highest order

$$\begin{aligned} (-ga_4 + a_1q^m)k - c_1f\bar{k} &= 0, \\ -\frac{c_2\bar{f}}{q}k + (-ga_2 + q^na_3)\bar{k} &= 0 \end{aligned}$$

hence,

$$c_1c_2f\bar{f} = q^{N+1}a_1a_3\left(\frac{a_4}{a_1q^m}g, \frac{a_2}{a_3q^n}g\right)_1. \quad (15)$$

By checking the lowest order similarly, we have

$$c_1c_2 = q^{N+1}\left(g, \frac{g}{q^{N+1}}\right)_1. \quad (16)$$

From (15) and (16), we get

$$f\bar{f} = \frac{a_1a_3\left(\frac{a_4}{a_1q^m}g, \frac{a_2}{a_3q^n}g\right)_1}{\left(g, \frac{g}{q^{N+1}}\right)_1}. \quad (17)$$

□

### 3 The case of $E_6^{(1)}$

In this section, we will derive the Lax pair for  $E_6^{(1)}$   $q$ -Painlevé equation (Theorem 3.1), the  $q$ -Painlevé equation (Theorem 3.2) and its special solutions in a determinant formula (Theorem 3.6).

#### 3.1 Construction of the difference equation

Let  $a_1, \dots, a_4 \in \mathbb{C} \setminus \{0\}$  be complex parameters. Define a function  $Y(x)$  as

$$Y(x) = \frac{(a_1x, a_2x, a_3, a_4)_\infty}{(a_1, a_2, a_3x, a_4x)_\infty}. \quad (18)$$

When

$$x_i := q^i, \quad (19)$$

we get

$$y_i := Y(x_i) = \frac{(a_3, a_4)_i}{(a_1, a_2)_i}. \quad (20)$$

Interpolate  $Y(x)$  by rational function

$$Y(x) = \frac{P_m(x)}{Q_n(x)} \quad (x = q^i, i = 0, 1, \dots, N) \quad (21)$$

where  $P_m(x)$  and  $Q_n(x)$  are the same form in section 2. We define the transformation  $T$  as

$$T : m \mapsto m - 1, \quad a_2 \mapsto qa_2. \quad (22)$$

We will consider the equations  $L_2, L_3$  in the equations (3) and (4).

**Theorem 3.1** *The explicit form of the equations  $L_2$  and  $L_3$  are given as*

$$L_2 : \left( \frac{x}{g} \right)_1 y(x) - (a_1 x)_1 y(qx) + c_1 x(xf)_1 \overline{y}(x) = 0, \quad (23)$$

$$L_3 : c_2 x \left( \frac{x\overline{f}}{q} \right)_1 y(x) + \left( a_2 x, \frac{x}{q^N}, \frac{x}{qg} \right)_1 \overline{y}(x) - \left( \frac{a_3}{q} x, \frac{a_4}{q} x, x \right)_1 \overline{y} \left( \frac{x}{q} \right) = 0. \quad (24)$$

**Proof**

First, we will consider the equation  $L_2$ . By definition,

$$\frac{Y(qx)}{Y(x)} = \frac{(a_3 x, a_4 x)_1}{(a_1 x, a_2 x)_1}, \quad \frac{\overline{Y}(x)}{Y(x)} = \frac{(a_2)_1}{(a_2 x)_1}.$$

The coefficient of  $\overline{y}(x)$  is same form in  $D_5^{(1)}$ . So,

$$\begin{aligned} Y(qx)P_m(x)Q_n(qx) - Y(x)P_m(qx)Q_n(x) &= \frac{Y(x)K(x)}{(a_1 x, a_2 x)_1}, \\ K(x) &= (a_3 x, a_4 x)_1 P_m(x)Q_n(qx) - (a_1 x, a_2 x)_1 P_m(qx)Q_n(x) \end{aligned}$$

where  $K(x)$  is a polynomial of  $x$  of order  $N + 2$ .

By definition for Padé interpolation, The polynomial  $K(x)$  have simple zeroes at  $x = 0$  and  $x = q^i$  ( $i = 0, \dots, N - 1$ ). So, the coefficient of  $\overline{y}(x)$  is given by

$$A'(x) := Y(qx)P_m(x)Q_n(qx) - Y(x)P_m(qx)Q_n(x) = \frac{c_0 x Y(x)}{(a_1 x, a_2 x)_1} \prod_{i=0}^{N-1} \left( \frac{x}{q^i} \right)_1 (xf)_1 \quad (25)$$

where  $c_0$  and  $f$  are some constants. Similarly, the coefficients of  $y(qx)$  and  $y(x)$  are given by

$$\begin{aligned} B'(x) &:= \overline{Y}(x)P_m(x)\overline{Q}_n(x) - Y(x)\overline{P}_m(x)Q_n(x) = \frac{cY(x)}{(a_2 x)_1} \prod_{i=0}^{N-1} \left( \frac{x}{q^i} \right)_1, \\ C'(x) &:= \overline{Y}(x)P_m(qx)Q_n(x) - Y(qx)\overline{P}_m(x)Q_n(qx) = \frac{c'Y(x)}{(a_1 x, a_2 x)_1} \prod_{i=0}^{N-1} \left( \frac{x}{q^i} \right)_1 \left( \frac{x}{g} \right)_1 \end{aligned} \quad (26)$$

where  $c$ ,  $c'$  and  $g$  are some constants. So, we get

$$c' \left( \frac{x}{g} \right)_1 y(x) - c(a_1 x)_1 y(qx) + c_0 x(xf)_1 \overline{y}(x) = 0. \quad (27)$$

We substitute  $x = 0$ . Then we get  $c' = c$ . Finally, by putting  $c_1 = c_0/c$ , we obtain  $L_2$ .

Next, we will consider the equation  $L_3$ . Similarly in  $D_5^{(1)}$ , the coefficients of  $y(x)$ ,  $\overline{y}(x)$  and  $\overline{y}(x/q)$  are given by  $-\overline{A}'(x/q)$ ,  $C'(x/q)$  and  $B'(x)$  respectively. So, we obtain  $L_3$ .  $\square$

### 3.2 Compatibility condition

We will consider the compatibility condition of the equations (23) and (24).

**Theorem 3.2** *The compatibility condition of the equations (23) and (24) is given by*

$$\left(\frac{1}{fg}, \frac{1}{f\bar{g}}\right)_1 = \frac{\left(\frac{a_1}{f}, \frac{q}{f}, \frac{a_3}{f}, \frac{a_4}{f}\right)_1}{\left(\frac{a_2}{f}, \frac{1}{q^N f}\right)_1}, \quad (28)$$

$$\frac{(fg, \bar{f}\bar{g})_1}{f\bar{f}} = \frac{q^{N-1}(a_1g, qg, a_3g, a_4g)_1}{a_2\left(a_1q^m g, \frac{a_3a_4q^n}{a_2}g\right)_1}. \quad (29)$$

*This is a  $q$ -Painlevé equation type  $E_6^{(1)}$ .*

**Proof**

First, we will prove the equation (28). We put  $x = 1/f$  in  $L_2$ .

$$\left(\frac{1}{fg}\right)_1 y\left(\frac{1}{f}\right) - \left(\frac{a_1}{f}\right)_1 y\left(\frac{q}{f}\right) = 0. \quad (30)$$

On the other hand, Translate  $L_3$  by  $T^{-1}$ , we have

$$\underline{L}_3 : \underline{c}_2 x \left(\frac{xf}{q}\right)_1 y(x) + \left(\frac{a_2}{q}x, \frac{x}{q^{N+1}}, \frac{x}{q\bar{g}}\right)_1 y(x) - \left(\frac{a_3}{q}x, \frac{a_4}{q}x, x\right)_1 y\left(\frac{x}{q}\right) = 0.$$

Putting  $x = q/f$  in  $\underline{L}_3$ , we have

$$\left(\frac{a_2}{f}, \frac{1}{q^N f}, \frac{1}{f\bar{g}}\right)_1 y\left(\frac{q}{f}\right) - \left(\frac{a_3}{f}, \frac{a_4}{f}, \frac{q}{f}\right)_1 y\left(\frac{1}{f}\right) = 0. \quad (31)$$

Hence,

$$\left(\frac{1}{fg}, \frac{1}{f\bar{g}}\right)_1 = \frac{\left(\frac{a_1}{f}, \frac{q}{f}, \frac{a_3}{f}, \frac{a_4}{f}\right)_1}{\left(\frac{a_2}{f}, \frac{1}{q^N f}\right)_1}. \quad (32)$$

Then, we prove the equation (29). We put  $x = g$  in  $L_2$ , we have

$$-(a_1g)y(qg) + c_1g(fg)_1\bar{y}(g) = 0.$$

On the other hand, putting  $x = qg$  in  $L_3$ , we have

$$c_2qg(\bar{f}g)_1y(qg) - (a_3g, a_4g, qg)_1\bar{y}(g) = 0.$$

Hence,

$$c_1c_2qg^2(fg, \bar{f}g)_1 = (a_1g, a_3g, a_4g, qg)_1. \quad (33)$$

We substitute for  $P(x) = \dots + kx^m$  in the  $L_2$  and  $L_3$  and check the highest order.

$$\left(a_1q^m - \frac{1}{g}\right)k - c_1f\bar{k} = 0, \quad (34)$$

$$\frac{c_2\bar{f}}{q}k + \left(\frac{a_2}{q^{N+1}g} - \frac{a_3a_4}{q^{m+1}}\right)\bar{k} = 0. \quad (35)$$

Then, we have

$$c_1 c_2 = \frac{a_2 \left( a_1 q^m g, \frac{a_3 a_4 q^n}{a_2} g \right)_1}{q^N g^2 f \bar{f}}, \quad (36)$$

From (33) and (36), we get

$$\frac{(fg, \bar{f}g)_1}{f \bar{f}} = \frac{q^{N-1} (a_1 g, a_3 g, a_4 g, qg)_1}{a_2 \left( a_1 q^m g, \frac{a_3 a_4 q^n}{a_2} g \right)_1}. \quad (37)$$

□

### 3.3 Explicit form for $f$ and $g$

We will derive the concrete form of  $P_m$  and  $Q_n$  (Lemma 3.4). Then, by using these formulas, we will derive the explicit form of  $f$  and  $g$  (Theorem 3.6).

First, we consider the Padé problem (21) for general  $\{x_i\}$  and  $\{y_i\}$ . We will prove following theorem.

**Theorem 3.3** *The polynomials  $P_m(x)$  and  $Q_n(x)$  are given as*

$$P_m(x) = F(x) \det \left( \sum_{s=0}^N x_s^{i+j} \frac{u_s}{x - x_s} \right)_{i,j=0}^n, \quad (38)$$

$$Q_n(x) = \det \left( \sum_{s=0}^N x_s^{i+j} u_s (x - x_s) \right)_{i,j=0}^{n-1} \quad (39)$$

where  $u_s = y_s / F'(x_s)$ ,  $F(x) = \prod_{i=0}^N (x - x_i)$ .

#### Proof

In this proof, we use the notation  $[k] = \{0, 1, \dots, k\}$  for  $k \in \mathbb{Z}_{\geq 0}$  and  $\Delta_{[k]} = \prod_{\substack{\alpha, \beta \in [k] \\ \alpha < \beta}} (x_\alpha - x_\beta)$ .

The determinant in the equation (38) is evaluated as

$$\begin{aligned} \det \left( \sum_{s=0}^N x_s^{i+j} \frac{u_s}{x - x_s} \right)_{i,j=0}^n &= \left| (x_s^i)_{\substack{i \in [n] \\ s \in [N]}} \text{diag} \left( \frac{u_s}{x - x_s} \right)_{s=0}^N (x_s^j)_{\substack{j \in [n] \\ s \in [N]}} \right| \\ &= \sum_{\substack{I \subset [N] \\ |I|=n+1}} \prod_{s \in I} \frac{u_s}{x - x_s} \left| (x_s^i)_{\substack{i \in [n] \\ s \in I}} \right| \left| (x_s^j)_{\substack{j \in [n] \\ s \in I}} \right| \\ &= \sum_{\substack{I \subset [N] \\ |I|=n+1}} \prod_{s \in I} \frac{u_s}{x - x_s} \Delta_I^2. \end{aligned}$$

Similarly,

$$\det \left( \sum_{s=0}^N x_s^{i+j} u_s (x - x_s) \right)_{i,j=0}^{n-1} = \sum_{\substack{I \subset [N] \\ |I|=n}} \Delta_I^2 \prod_{s \in I} u_s (x - x_s).$$



Let

$$R(x) = \sum_{\substack{I \subset [N] \\ |I|=n+1}} \Delta_I^2 \prod_{s \in I} \frac{u_s}{x - x_s}, \quad W(x) = \sum_{\substack{I \subset [N] \\ |I|=n}} \Delta_I^2 \prod_{s \in I} u_s (x - x_s)$$

and we will prove

$$\left. \frac{F(x)R(x)}{W(x)} \right|_{x=x_i} = y_i. \quad (40)$$

We substitute  $x = x_i (i \in [N])$  in  $F(x)R(x)$  and put  $I = I' \cup \{i\}$ , then

$$\begin{aligned} F(x_i)R(x_i) &= \sum_{\substack{I \subset [N] \\ |I|=n+1}} \Delta_I^2 \prod_{s \in I} u_s \prod_{s \notin I} (x_i - x_s) \\ &= \sum_{\substack{I' \cup \{i\} \subset [N] \\ |I' \cup \{i\}|=n+1}} \Delta_{I' \cup \{i\}}^2 \prod_{s \in I' \cup \{i\}} u_s \prod_{s \notin I' \cup \{i\}} (x_i - x_s) \\ &= \sum_{\substack{I' \cup \{i\} \subset [N] \\ |I'|=n}} \Delta_{I' \cup \{i\}}^2 u_i \prod_{s \in I'} u_s \prod_{\substack{s \notin I' \\ s \neq i}} (x_i - x_s) \\ &= u_i \sum_{\substack{I' \cup \{i\} \subset [N] \\ |I'|=n}} \Delta_{I'}^2 \prod_{s \in I'} (x_i - x_s)^2 \prod_{\substack{s \in I' \\ s \neq i}} u_s \prod_{s \notin I'} (x_i - x_s) \\ &= u_i F'(x_i) \sum_{\substack{I' \cup \{i\} \subset [N] \\ |I'|=n}} \Delta_{I'}^2 \prod_{s \in I'} (x_i - x_s) \prod_{s \in I'} u_s, \\ W(x_i) &= \sum_{\substack{I \subset [N] \\ |I|=n}} \Delta_I^2 \prod_{s \in I} u_s (x_i - x_s) \\ &= \sum_{\substack{I' \subset [N] \\ |I'|=n}} \Delta_{I'}^2 \prod_{s \in I'} u_s (x_i - x_s) \quad (\text{if } s = i, \prod_{s \in I} u_s (x_i - x_s) = 0). \end{aligned}$$

So,

$$\frac{F(x_i)R(x_i)}{W(x_i)} = u_i F'(x_i) = y_i.$$

Hence, (40) is proved.  $\square$

**Lemma 3.4** *By specializing  $x_i$  and  $y_i$  as the equations (19) and (20) respectively, we have*

$$P_m(x) = F(x) \det \left( \frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(a_1, a_2, q)_s} \frac{q^{s(i+j+1)}}{x - q^s} \right)_{i,j=0}^n, \quad (41)$$

$$Q_n(x) = \det \left( \frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(a_1, a_2, q)_s} q^{s(i+j+1)} (x - q^s) \right)_{i,j=0}^{n-1}. \quad (42)$$

**Proof**

First, we calculate  $F'(x_s)$ .

$$\begin{aligned} F'(x_s) &= (x_s - x_0) \cdots (x_s - x_{s-1})(x_s - x_{s+1}) \cdots (x_s - x_N) \\ &= (-1)^s q^{1/2(s-1)s} (q)_s q^{s(N-s)} (q)_{N-s}. \end{aligned}$$

Here,

$$\begin{aligned} (q)_{N-s} &= \frac{(q)_\infty}{(q^{N-s+1})_\infty} \\ &= \frac{(q)_\infty}{(q^{N-s+1})_s (q^{N+1})_\infty} \end{aligned}$$

Furthermore,

$$(q^{-N})_s = (-1)^s q^{(s-1)s/2 - Ns} (q^{N-s+1})_s. \quad (43)$$

So,

$$(q)_{N-s} = (-1)^s q^{s(s-1)/2 - Ns} \frac{(q)_\infty}{(q^{-N})_s (q^{N+1})_\infty}. \quad (44)$$

By using this relation (44),  $F'(x)$  is given by

$$F'(x_s) = q^{-s} \frac{(q)_s (q)_\infty}{(q^{-N})_s (q^{N+1})_\infty}. \quad (45)$$

By using the equation (45), the equation (41) is evaluated as

$$\begin{aligned} P_m(x) &= F(x) \det \left( \sum_{s=0}^N x_s^{i+j} \frac{u_s}{x - x_s} \right)_{i,j=0}^n \\ &= F(x) \det \left( \frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(a_1, a_2, q)_s} \frac{q^{s(i+j+1)}}{x - q^s} \right)_{i,j=0}^n. \end{aligned}$$

$Q_n(x)$  is similar. □

By using the Lemma 3.4, we get the following lemma.

**Lemma 3.5** *The values of polynomials  $P_m(x)$  and  $Q_n(x)$  at special points are given as follows.*

$$\begin{aligned} P_m\left(\frac{1}{a_1}\right) &= \frac{(a_1)_{N+1}}{a_1^m} \det \left( \frac{(q^{N+1})_\infty}{(a_1)_1 (q)_\infty} {}_3\varphi_2 \left( \begin{matrix} a_3, a_4, q^{-N} \\ q a_1, a_2 \end{matrix}; q^{i+j+1} \right) \right)_{i,j=0}^n, \\ P_m\left(\frac{1}{a_2}\right) &= \frac{(a_2)_{N+1}}{a_2^m} \det \left( \frac{(q^{N+1})_\infty}{(a_2)_1 (q)_\infty} {}_3\varphi_2 \left( \begin{matrix} a_3, a_4, q^{-N} \\ a_1, q a_2 \end{matrix}; q^{i+j+1} \right) \right)_{i,j=0}^n, \\ P_m\left(\frac{q}{a_3}\right) &= \left(\frac{q}{a_3}\right)^m \left(\frac{a_3}{q}\right)_{N+1} \det \left( \frac{(q^{N+1})_\infty}{(a_3/q)_1 (q)_\infty} {}_3\varphi_2 \left( \begin{matrix} a_3/q, a_4, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1} \right) \right)_{i,j=0}^n, \\ P_m\left(\frac{q}{a_4}\right) &= \left(\frac{q}{a_4}\right)^m \left(\frac{a_4}{q}\right)_{N+1} \det \left( \frac{(q^{N+1})_\infty}{(a_4/q)_1 (q)_\infty} {}_3\varphi_2 \left( \begin{matrix} a_3, a_4/q, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1} \right) \right)_{i,j=0}^n, \\ Q_n\left(\frac{q}{a_1}\right) &= \left(\frac{q}{a_1}\right)^n \det \left( \left(\frac{a_1}{q}\right)_1 \frac{(q^{N+1})_\infty}{(q)_\infty} {}_3\varphi_2 \left( \begin{matrix} a_3, a_4, q^{-N} \\ a_1/q, a_2 \end{matrix}; q^{i+j+1} \right) \right)_{i,j=0}^{n-1}, \end{aligned}$$

$$\begin{aligned}
Q_n\left(\frac{q}{a_2}\right) &= \left(\frac{q}{a_2}\right)^n \det\left(\left(\frac{a_2}{q}\right)_1 \frac{(q^{N+1})_\infty}{(q)_\infty} {}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ a_1, a_2/q \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}, \\
Q_n\left(\frac{1}{a_3}\right) &= \left(\frac{1}{a_3}\right)^n \det\left((a_3)_1 \frac{(q^{N+1})_\infty}{(q)_\infty} {}_3\varphi_2\left(\begin{matrix} qa_3, a_4, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}, \\
Q_n\left(\frac{1}{a_4}\right) &= \left(\frac{1}{a_4}\right)^n \det\left((a_4)_1 \frac{(q^{N+1})_\infty}{(q)_\infty} {}_3\varphi_2\left(\begin{matrix} a_3, qa_4, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}
\end{aligned}$$

where

$${}_3\varphi_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right) = \sum_{s=0}^{\infty} \frac{(a_1, a_2, a_3)_s}{(b_1, b_2, q)_s} x^s. \quad (46)$$

${}_3\varphi_2$  is a  $q$ -hypergeometric series.

### Proof

We prove  $P_m(1/a_1)$ . Using the equation (41),

$$\begin{aligned}
P_m\left(\frac{1}{a_1}\right) &= F\left(\frac{1}{a_1}\right) \det\left(\frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(a_1, a_2, q)_s} \frac{q^{s(i+j+1)}}{1/a_1 - q^s}\right)_{i,j=0}^n \\
&= \prod_{i=0}^N \left(\frac{1}{a_1} - q^i\right) \det\left(\frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(a_1, a_2, q)_s} \frac{q^{s(i+j+1)}}{1 - a_1 q^s}\right)_{i,j=0}^n \\
&= \left(\frac{1}{a_1}\right)^{N+1} \prod_{i=0}^N (1 - q^i a_1) \det\left(\frac{a_1}{1 - a_1} \frac{(q^{N+1})_\infty}{(q)_\infty} \sum_{s=0}^N \frac{(a_3, a_4, q^{-N})_s}{(qa_1, a_2, q)_s} q^{s(i+j+1)}\right)_{i,j=0}^n \\
&= \frac{(a_1)_{N+1}}{a_1^m} \det\left(\frac{(q^{N+1})_\infty}{(a_1)_1 (q)_\infty} {}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ qa_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^n.
\end{aligned}$$

Others are similar. □

By using the above Lemma 3.5, we will get the following theorem.

**Theorem 3.6** *The explicit form of  $f$  and  $g$  are*

$$\frac{\left(\frac{f}{a_1}\right)_1}{\left(\frac{f}{a_2}\right)_1} = A \frac{\det\left({}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ qa_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^n \det\left({}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ a_1/q, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}}{\det\left({}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ a_1, qa_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^n \det\left({}_3\varphi_2\left(\begin{matrix} a_3, a_4, q^{-N} \\ a_1, a_2/q \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}}, \quad (47)$$

$$\frac{(ga_3)_1}{(ga_4)_1} = B \frac{\det\left({}_3\varphi_2\left(\begin{matrix} a_3/q, a_4, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^n \det\left({}_3\varphi_2\left(\begin{matrix} qa_3, a_4, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}}{\det\left({}_3\varphi_2\left(\begin{matrix} a_3, a_4/q, q^{-N} \\ a_1, a_2 \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^n \det\left({}_3\varphi_2\left(\begin{matrix} a_3, qa_4, q^{-N} \\ a_1, a_2, q \end{matrix}; q^{i+j+1}\right)\right)_{i,j=0}^{n-1}} \quad (48)$$

where

$$A = \frac{a_1}{a_2} \frac{\left( \frac{a_3}{a_1}, \frac{a_4}{a_1}, \overbrace{a_2, \dots, a_2}^{n+1}, \overbrace{\frac{a_1}{q}, \dots, \frac{a_1}{q}}^n, q^N a_1 \right)_1}{\left( \frac{a_3}{a_2}, \frac{a_4}{a_2}, \overbrace{a_1, \dots, a_1}^{n+1}, \overbrace{\frac{a_2}{q}, \dots, \frac{a_2}{q}}^n, q^N a_2 \right)_1},$$

$$B = \frac{a_4}{a_3} \frac{\left( \frac{a_1}{a_3}, \overbrace{\frac{a_4}{q}, \dots, \frac{a_4}{q}}^{n+1}, \overbrace{a_3, \dots, a_3}^n, \frac{a_3}{q} \right)_1}{\left( \frac{a_1}{a_4}, \overbrace{\frac{a_3}{q}, \dots, \frac{a_3}{q}}^{n+1}, \overbrace{a_4, \dots, a_4}^n, \frac{a_4}{q} \right)_1}.$$

### Proof

First, we prove the equation (47). We substitute  $x = 1/a_1, 1/a_2$  for the equation (25), we have

$$\left( \frac{a_3}{a_1}, \frac{a_4}{a_1} \right)_1 P_m \left( \frac{1}{a_1} \right) Q_n \left( \frac{q}{a_1} \right) = \prod_{i=0}^{N-1} \left( \frac{1}{q^i a_1} \right)_1 \frac{c_0}{a_1} \left( \frac{f}{a_1} \right)_1,$$

$$\left( \frac{a_3}{a_2}, \frac{a_4}{a_2} \right)_1 P_m \left( \frac{1}{a_2} \right) Q_n \left( \frac{q}{a_2} \right) = \prod_{i=0}^{N-1} \left( \frac{1}{q^i a_2} \right)_1 \frac{c_0}{a_2} \left( \frac{f}{a_2} \right)_1$$

respectively. Taking a ratio of these two equations, we have

$$\frac{\left( \frac{f}{a_1} \right)_1}{\left( \frac{f}{a_2} \right)_1} = \left( \frac{a_1}{a_2} \right)^{N+1} \frac{\left( \frac{a_3}{a_1}, \frac{a_4}{a_1} \right)_1}{\left( \frac{a_3}{a_2}, \frac{a_4}{a_2} \right)_1} \prod_{i=0}^{N-1} \frac{(q^i a_2)_1}{(q^i a_1)_1} \frac{P_m \left( \frac{1}{a_1} \right) Q_n \left( \frac{q}{a_1} \right)}{P_m \left( \frac{1}{a_2} \right) Q_n \left( \frac{q}{a_2} \right)}.$$

Then, by using the Lemma 3.5, we obtain the equation (47).

The proof of the equation (48) is similar, where we substitute  $x = 1/a_3, 1/a_4$  for the equation (26).  $\square$

## A Relation to the QRT system

Painlevé equation is a non-autonomization of the QRT system[11]. In this appendix, Using a pencil (i.e. a 1-parameter family) on  $\mathbb{P}^1 \times \mathbb{P}^1$  of order (2,2), we will derive the autonomization of the equations (10), (11) for  $q$ -Painlevé VI equation and the equations (28), (29) for  $q$ -Painlevé equation of type  $E_6^{(1)}$  from the QRT point of view.

### A.1 The case of $q$ -Painlevé VI equation

We consider a polynomial of order (2,2) passing the eight points at

$$(a_1, 0), (a_2, 0), (0, a_3), (0, a_4), (a_5, \infty), (a_6, \infty), (\infty, a_7), (\infty, a_8) \quad (49)$$

(Figure 1). Then, the following lemma is obtained.

**Lemma A.1** *When the eight parameters  $a_1, \dots, a_8$  satisfy the condition*

$$\frac{a_1 a_2 a_7 a_8}{a_3 a_4 a_5 a_6} = 1, \quad (50)$$

*the polynomial of order  $(2, 2)$  passing the eight points (49) forms a 1-parameter family.*

**Proof**

Let  $P(x)$  be a polynomial of order  $(2, 2)$  passing through the eight points (49). First, we consider the conditions that  $P(x)$  passes the points at  $(a_1, 0)$ ,  $(a_2, 0)$ ,  $(0, a_3)$ ,  $(0, a_4)$ ,  $(a_5, \infty)$ ,  $(a_6, \infty)$ . Then,  $P(x)$  is determined as

$$x^2 - (a_1 + a_2)x + a_1 a_2 - \frac{a_1 a_2}{a_3 a_4}(a_3 + a_4)y + \frac{a_1 a_2}{a_3 a_4}y^2 - \frac{a_1 a_2}{a_3 a_4 a_5 a_6}(a_5 + a_6)xy^2 + \frac{a_1 a_2}{a_3 a_4 a_5 a_6}x^2 y^2 + ax^2 y + \lambda xy = 0 \quad (51)$$

up to over all constant, where  $a$  and  $\lambda$  are parameters. We substitute  $x = 1/u$  for the equation (51) and put  $u = 0$ . Then,

$$\frac{a_1 a_2}{a_3 a_4 a_5 a_6}y^2 + ay + 1 = 0. \quad (52)$$

By definition of the polynomial  $P(x)$ , the solutions of the equation (52) are  $a_7$  and  $a_8$ . So,

$$\frac{a_1 a_2}{a_3 a_4 a_5 a_6}y^2 + ay + 1 = \frac{a_1 a_2}{a_3 a_4 a_5 a_6}(y - a_7)(y - a_8). \quad (53)$$

By comparing coefficients of the equation above, we get the condition (50) and the relation

$$a = -\frac{a_1 a_2(a_7 + a_8)}{a_3 a_4 a_5 a_6}. \quad (54)$$

As a result,  $P(x)$  is given by

$$\lambda xy + F(x, y) = 0 \quad (55)$$

where  $F(x, y)$  is a polynomial of order  $(2, 2)$  and  $\lambda$  is a parameter.  $\square$

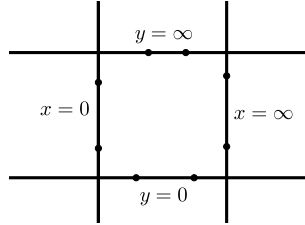


Figure 1: Eight points for  $q$ -Painlevé VI

We consider the case where the condition (50) is satisfied. When we give a generic initial point  $(x_0, y_0)$ , then the curve  $C_0$  passing through it is determined uniquely. Then we get the following theorem.

**Theorem A.2** Consider the intersection of the curve  $C_0$  and  $\{x = x_0\}$ , and let  $(x_0, y_0)$  and  $(x_0, y_1)$  be the intersection points. Then, we get

$$y_0 y_1 = \frac{(x_0 - a_1)(x_0 - a_2)}{(x_0 - a_5)(x_0 - a_6)} a_7 a_8, \quad (56)$$

Similarly, Consider the intersection of  $C_0$  and  $\{y = y_0\}$ , and let  $(x_0, y_0)$  and  $(x_1, y_0)$  be the intersection points. Then,

$$x_0 x_1 = \frac{(y_0 - a_3)(y_0 - a_4)}{(y_0 - a_5)(y_0 - a_6)} a_5 a_6. \quad (57)$$

**Proof**

We prove the equation (56). First, the curve  $C_0$  is given by the equation (55) where  $\lambda = -F(x_0, y_0)/x_0 y_0$ . Let  $F(x, y) = A(x)y^2 + B(x)y + C(x)$  where  $A(x)$ ,  $B(x)$  and  $C(x)$  are second degree polynomials with respect to  $x$ . And we put  $x = x_0$  in the curve  $C_0$ . Then we get a following equation

$$A(x_0)y_0 y^2 - \{A(x_0)y_0^2 + C(x_0)\}y + C(x_0)y_0 = 0. \quad (58)$$

The solutions of the equation (58) are  $y_0$  and  $y_1$ . So, by the relation of roots and coefficients, we get

$$y_0 y_1 = \frac{C(x_0)}{A(x_0)}. \quad (59)$$

Then, by the definition for the polynomial  $F(x, y)$ , it follows that

$$F(a_i, 0) = 0 \quad (i = 1, 2), \quad F(a_j, \infty) = 0 \quad (j = 5, 6).$$

And the solutions of the equation  $F(0, y) = 0$  are  $a_3, a_4$ . So, by the relation of roots and coefficients,  $y_0 y_1 = a_3 a_4$  when  $x_0 = 0$ .

By the results of above and the condition (50), we get the equation (56). The equation (57) is similar.  $\square$

The equations (56) and (57) are the same form for the  $q$ -Painlevé VI equation (10) and (11).

## A.2 The case of $E_6^{(1)}$

Similarly in appendix A.1, we consider the polynomial of order (2, 2) and eight points at

$$(a_1, 0), (a_2, 0), (0, a_3), (0, a_4), (a_5, \frac{1}{a_5}), (a_6, \frac{1}{a_6}), (a_7, \frac{1}{a_7}), (a_8, \frac{1}{a_8}) \quad (60)$$

(Figure 2). Then, the following lemma is obtained.

**Lemma A.3** When the eight parameters  $a_1, a_2, \dots, a_8$  satisfy the condition

$$\frac{a_3 a_4 a_5 a_6 a_7 a_8}{a_1 a_2} = 1, \quad (61)$$

the polynomial of order (2, 2) passing the eight points (60) forms a 1-parameter family.

**Proof**

Let  $P(x)$  be a polynomial of order  $(2, 2)$  passing through the eight points (49). First, we consider the conditions that  $P(x)$  passes the points at  $(a_1, 0)$ ,  $(a_2, 0)$ ,  $(0, a_3)$ ,  $(0, a_4)$ . Then,  $P(x)$  is determined as

$$x^2 - (a_1 + a_2)x + a_1a_2 - \frac{a_1a_2}{a_3a_4}(a_3 + a_4)y + \frac{a_1a_2}{a_3a_4}y^2 + ax^2y^2 + bx^2y + cxy^2 + \lambda xy = 0 \quad (62)$$

up to over all constant, where  $a, b, c$  and  $\lambda$  are parameters. We substitute  $x = u$  and  $y = 1/u$ . Then,

$$u^4 + (b - a_1 - a_2)u^3 + (a_1a_2 + a + \lambda)u^2 + \left\{c - \frac{a_1a_2}{a_3a_4}(a_3 + a_4)\right\}u + \frac{a_1a_2}{a_3a_4} = 0 \quad (63)$$

By definition of the polynomial  $P(x)$ , the solutions of the equation (63) are  $a_5, a_6, a_7$  and  $a_8$ . So,

$$\begin{aligned} u^4 + (b - a_1 - a_2)u^3 + (a_1a_2 + a + \lambda)u^2 + \left\{c - \frac{a_1a_2}{a_3a_4}(a_3 + a_4)\right\}u + \frac{a_1a_2}{a_3a_4} \\ = (u - a_5)(u - a_6)(u - a_7)(u - a_8). \end{aligned} \quad (64)$$

By comparing coefficients of the equation above, we get the condition (61) and the following relations

$$\begin{aligned} a &= -\lambda - a_1a_2 + a_5a_6 + a_5a_7 + a_5a_8 + a_6a_7 + a_6a_8 + a_7a_8, \\ b &= a_1 + a_2 - (a_5 + a_6 + a_7 + a_8), \\ c &= \frac{a_1a_2}{a_3a_4}(a_3 + a_4) - (a_5a_6a_7 + a_5a_6a_8 + a_5a_7a_8 + a_6a_7a_8). \end{aligned}$$

As a result,  $P(x)$  is given by

$$\lambda xy(1 - xy) + F(x, y) = 0 \quad (65)$$

where  $F(x, y)$  is a polynomial of order  $(2, 2)$  and  $\lambda$  is a parameter.  $\square$

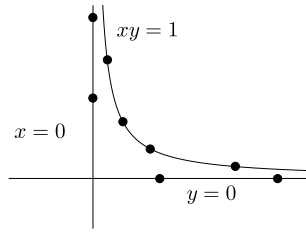


Figure 2: Eight points for  $E_6^{(1)}$

We consider the case where the condition (61) is satisfied. We give a generic initial point  $(x_0, y_0)$ , then the curve  $C_0$  passing through it is determined uniquely. Then we get the following theorem.

**Theorem A.4** Consider the intersection of the curve  $C_0$  and  $\{x = x_0\}$ , and let  $(x_0, y_0)$  and  $(x_0, y_1)$  be intersection points. Then, we get

$$\frac{(x_0 y_1 - 1)(x_0 y_0 - 1)}{y_0 y_1} = \frac{(x_0 - a_5)(x_0 - a_6)(x_0 - a_7)(x_0 - a_8)}{(x_0 - a_1)(x_0 - a_2)}. \quad (66)$$

Similarly, Consider the intersection of the curve  $C_0$  and  $\{y = y_0\}$ , and let  $(x_0, y_0)$  and  $(x_1, y_0)$  be intersection points. Then,

$$\frac{(x_1 y_0 - 1)(x_0 y_0 - 1)}{x_0 x_1} = \frac{(y_0 a_5 - 1)(y_0 a_6 - 1)(y_0 a_7 - 1)(y_0 a_8 - 1)}{(y_0 - a_3)(y_0 - a_4)}. \quad (67)$$

**Proof**

We prove the equation (66). First, the curve  $C_0$  is given by the equation (65) where  $\lambda = -F(x_0, y_0)/x_0 y_0(1 - x_0 y_0)$ . Let  $F(x, y) = A(x)y^2 + B(x)y + C(x)$  where  $A(x)$ ,  $B(x)$  and  $C(x)$  are second degree polynomials with respect to  $x$ . And we put  $x = x_0$  in the curve  $C_0$ . Then, we get the following equation

$$(A(x_0)y_0 + B(x_0)x_0 y_0 + C(x_0)x_0)y^2 - (A(x_0)y_0^2 + B(x_0)x_0 y_0^2 + C(x_0))y + C(x_0)y_0(1 - x_0 y_0) = 0. \quad (68)$$

The solutions of the equation (68) are  $y_0$  and  $y_1$ . So, by the relation of roots and coefficient, we get

$$y_0 y_1 = \frac{C(x_0)y_0(1 - x_0 y_0)}{A(x_0)y_0 + B(x_0)x_0 y_0 + C(x_0)x_0}. \quad (69)$$

Namely,

$$y_1 = \frac{C(x_0)(1 - x_0 y_0)}{A(x_0)y_0 + B(x_0)x_0 y_0 + C(x_0)x_0}. \quad (70)$$

We substitute the equation (70) for  $(x_0 y_1 - 1)/y_1$ . Then, we get the following equation

$$\begin{aligned} \frac{x_0 y_1 - 1}{y_1} &= -\frac{(A(x_0) + B(x_0)x_0 + C(x_0)x_0^2)y_0}{C(x_0)(1 - x_0 y_0)} \\ &= -\frac{x_0^2 F(x_0, 1/x_0)y_0}{C(x_0)(1 - x_0 y_0)}. \end{aligned} \quad (71)$$

The order of the numerator and the denominator of the r.h.s. of the equation (71) are  $(4, 1)$  and  $(3, 1)$  respectively. By the definition for the polynomial  $F(x, y)$ , it follows that

$$a_i^2 F(a_i, 1/a_i) = 0 \quad (i = 5, 6, 7, 8), \quad F(a_j, 0) = 0 \quad (j = 1, 2).$$

and hence,  $y_0 y_1 = a_3 a_4$  when  $x_0 = 0$ .

By the results of above and the condition (61), we get the equation (66). The equation (67) is similar.  $\square$

The equations (66) and (67) are same form for the  $q$ -Painlevé equation of type  $E_6^{(1)}$  (28) and (29).



## B Padé Method and Bäcklund transformations

In the main text, we constructed the  $q$ -Painlevé equation of type  $E_6^{(1)}$  with respect to the direction (22). But, we can construct  $q$ -Painlevé equation by using other directions.

In this appendix, we will discuss the relation between these  $q$ -Painlevé equations along the various direction and Bäcklund transformations. First, we will summarize about Bäcklund transformations. Then, we will discuss the relation between the  $q$ -Painlevé equations for various directions and Bäcklund transformations.

### B.1 Bäcklund transformations

We formulate the Bäcklund transformations of the affine Weyl group of type  $E_6^{(1)}$ .

Let  $A$  be the Cartan matrix of type  $E_6^{(1)}$ :

$$A = (a_{ij})_{i,j=0}^6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

We define the transformations  $s_i$  ( $i = 0, 1, 2, 3, 4, 5, 6$ ) and  $\pi_j$  ( $j = 1, 2$ ) on the parameters  $b_k$  ( $k = 1, 2, 3, 4, 5, 6, 7, 8$ ) and variables  $f$  and  $g$  as follows.

$$\begin{aligned} s_0 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto (b_1, b_2, b_4, b_3, b_5, b_6, b_7, b_8, f, g), \\ s_1 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto (b_2, b_1, b_3, b_4, b_5, b_6, b_7, b_8, f, g), \\ s_2 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto \left( b_1, \frac{1}{b_8}, b_2 b_3 b_8, b_2 b_4 b_8, b_5, b_6, \right. \\ &\quad \left. b_7, \frac{1}{b_2}, f, \frac{b_2(b_8 - f)g}{1 - b_2 f - f g + b_2 b_8 f g} \right) \\ s_3 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto (b_1, b_2, b_3, b_4, b_8, b_6, b_7, b_5, f, g), \\ s_4 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto (b_1, b_2, b_3, b_4, b_6, b_5, b_7, b_8, f, g), \\ s_5 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto (b_1, b_2, b_3, b_4, b_5, b_7, b_6, b_8, f, g), \\ s_6 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto \left( b_1 b_3 b_8, b_2 b_3 b_8, \frac{1}{b_8}, b_4, b_5, b_6, \right. \\ &\quad \left. b_7, \frac{1}{b_3}, \frac{f(b_8 g - 1)}{-b_3 b_8 + b_8 g - f g + b_3 b_8 f g}, g \right) \\ \pi_1 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto \left( \frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}, \frac{1}{b_5}, \frac{1}{b_6}, \frac{1}{b_7}, \frac{1}{b_8}, g, f \right), \\ \pi_2 : (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, f, g) &\mapsto \left( \frac{1}{b_7}, \frac{1}{b_6}, \frac{1}{b_3 b_5 b_8}, \frac{1}{b_4 b_5 b_8}, b_8, \frac{1}{b_2}, \right. \\ &\quad \left. \frac{1}{b_1}, b_5, f, \frac{1 - f g}{f + b_5 b_8 g - b_5 f g - b_8 f g} \right) \end{aligned} \tag{72}$$

**Remark B.1** The actions  $s_0, \dots, s_6, \pi_1$ , and  $\pi_2$  are invariant under the following transformation of

$$(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8; f, g) \mapsto (b_1/\lambda, b_2/\lambda, b_3/\lambda, b_4/\lambda, \lambda b_5, \lambda b_6, \lambda b_7, \lambda b_8; \lambda f, g/\lambda)$$

where  $\lambda$  is any parameter.

Then, we have the following lemma.

**Lemma B.2** The transformations (72) satisfy the fundamental relation of the affine Weyl group  $W(E_6^{(1)})$ :

$$\begin{aligned} s_i^2 &= 1 \quad (i = 0, \dots, 6), & \pi_i^2 &= 1 \quad (i = 1, 2), \\ (s_i s_j)^3 &= 1 \quad (a_{ij} = -1), & (s_i s_j)^2 &= 1 \quad (a_{ij} = 0), \\ (\pi_1 \pi_2)^3 &= 1, & & \\ s_i \pi_1 &= \pi_1 s_j & ((i, j) &= (1, 0), (2, 6), (3, 3), (4, 4), (5, 5)), \\ s_i \pi_2 &= \pi_2 s_j & ((i, j) &= (0, 0), (1, 5), (2, 4), (3, 3), (6, 6)), \end{aligned}$$

**Proof**

Direct calculation. □

We put  $q = b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8$ . It is well known that for any element  $T$  of the translation subgroup of  $W(E_6^{(1)})$ , the operation of  $T$  is regarded as Painlevé equation. For example, put

$$T = r' r \tag{73}$$

where

$$r' = \pi_1 r \pi_1, \tag{74}$$

$$r = \pi_2 s_0 s_5 s_4 s_5 s_3 s_4 s_5 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_5. \tag{75}$$

Then, the following theorem is obtained.

**Theorem B.3** For  $T \in W(E_6^{(1)})$  given the equation (73), we have

$$\left( \frac{1}{f \underline{g}}, \frac{1}{f \overline{g}} \right)_1 = \frac{\left( \frac{b_5}{f}, \frac{b_6}{f}, \frac{b_7}{f}, \frac{b_8}{f} \right)_1}{\left( \frac{1}{b_1 f}, \frac{1}{b_2 f} \right)_1} \tag{76}$$

$$\frac{(\overline{f} \underline{g}, f \overline{g})_1}{f \overline{f}} = \frac{b_1 b_2 (b_5 \underline{g}, b_6 \underline{g}, b_7 \underline{g}, b_8 \underline{g})_1}{q \left( \frac{\underline{g}}{b_3}, \frac{\underline{g}}{b_4} \right)_1} \tag{77}$$

where  $\underline{g} = T^{-1}(g) = r^{-1}(g)$ ,  $\overline{f} = T(f) = r'(f)$ . Furthermore by operating  $T$  to the parameters  $(b_1, \dots, b_8)$ , they shift as

$$(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8) \mapsto (b_1/q, b_2/q, qb_3, qb_4, b_5, b_6, b_7, b_8) \tag{78}$$

**Proof**

Direct computation by using

$$s_2 \left( \frac{1}{fg} \right)_1 = \frac{\left( \frac{1}{b_2 g} \right)_1}{\left( \frac{b_8}{f} \right)_1} \left( \frac{1}{fg} \right)_1, \quad \pi_2 \left( \frac{1}{fg} \right)_1 = \frac{\left( \frac{b_5}{f}, \frac{b_8}{f} \right)_1}{\left( \frac{1}{fg} \right)_1}.$$

Then, we obtain the equation (76). The equation (77) is similar. And the relation (78) is derived by operating  $T$  for the parameters  $(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ .  $\square$

## B.2 Painlevé equations along the various directions and Bäcklund transformations

By the Padé method, we can construct  $q$ -Painlevé equations along the various direction  $T_i$  of deformations. These equations are equivalent through some Bäcklund transformations of variables  $(f, g) \mapsto (f_i, g_i)$  where  $f$  and  $g$  are variables in section 3. The variable  $f$  is defined by the equation (25) which does not depend on the direction. So, the variable  $f$  does not depend on the direction. On the other hand, the variable  $g$  is defined by the equation (26) which depends on the direction. So, the variable  $g$  depends on the direction in general.

In this appendix, we will present the correspondence of variables  $g_i$  by using Bäcklund transformations.

For example, we take the following transformation  $T_1$  in the Padé method

$$T_1 : a_1 \mapsto qa_1, a_2 \mapsto qa_2, a_3 \mapsto qa_3, a_4 \mapsto qa_4$$

Then, we get the following theorem

**Theorem B.4** *For the direction  $T_1$ , we get*

$$\left( \frac{1}{f_1 g_1}, \frac{q}{f_1 \bar{g}_1} \right)_1 = \frac{\left( \frac{a_1}{f_1}, \frac{a_2}{f_1}, \frac{a_3}{f_1}, \frac{a_4}{f_1} \right)_1}{\left( \frac{1}{q^N f_1}, \frac{q}{f_1} \right)_1}, \quad (79)$$

$$\frac{\left( f_1 g_1, \frac{\bar{f}_1 g_1}{q} \right)_1}{f_1 \bar{f}_1} = \frac{q^{N-1} (a_1 g_1, a_2 g_1, a_3 g_1, a_4 g_1)_1}{(q^m a_1 a_2 g_1, q^n a_3 a_4 g_1)_1}. \quad (80)$$

**Proof**

By similar, calculation in section 3, we get the Lax equation  $L_2$  and  $L_3$  as

$$L_2 : (a_3 x, a_4 x)_1 y(x) - \left( \frac{x}{q^N}, \frac{x}{g_1} \right)_1 y(qx) + c_1 x (f_1 x)_1 \bar{y}(x) = 0, \quad (81)$$

$$L_3 : c_2 x \left( \frac{\bar{f}_1 x}{q} \right)_1 y(x) + (a_1 x, a_2 x)_1 \bar{y}(x) - \left( x, \frac{x}{g_1} \right)_1 \bar{y} \left( \frac{x}{q} \right) = 0 \quad (82)$$

where  $c_1$  and  $c_2$  are constants. Then we get the  $q$ -Painlevé equation (79) and (80).  $\square$

Here, we make a correspondence between the parameters in (79), (80) and the parameters in (76), (77). One of the possible choice is

$$b_1 = q^N, \quad b_2 = 1/q, \quad b_3 = \frac{1}{q^m a_1 a_2}, \quad b_4 = \frac{1}{q^n a_3 a_4}, \quad b_5 = a_1, \quad b_6 = a_2, \quad b_7 = a_3, \quad b_8 = a_4.$$

In the followings, we always identify the parameters  $m, n, a_1, \dots, a_4$  with  $b_1, \dots, b_8$ . by this rule. In particular the direction  $T_1$  above is  $T$  in the equation (73).

By using this correspondence,  $T_1$  is rewritten as

$$T_1 : b_3 \mapsto b_3/q^2, \quad b_4 \mapsto b_4/q^2, \quad b_5 \mapsto qb_5, \quad b_6 \mapsto qb_6, \quad b_7 \mapsto qb_7, \quad b_8 \mapsto qb_8.$$

And the Painlevé equation (79), (80) are rewritten as

$$\left( \frac{1}{f_1 g_1}, \frac{q}{f_1 \underline{g_1}} \right)_1 = \frac{\left( \frac{b_5}{f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}, \frac{b_8}{f_1} \right)_1}{\left( \frac{1}{b_1 f_1}, \frac{1}{b_2 f_1} \right)_1}, \quad \frac{\left( f_1 g_1, \frac{\overline{f_1 g_1}}{q} \right)_1}{f_1 \overline{f_1}} = \frac{b_1 b_2 (b_5 g_1, b_6 g_1, b_7 g_1, b_8 g_1)_1}{\left( \frac{g_1}{b_3}, \frac{g_1}{b_4} \right)_1}. \quad (83)$$

By using transformation in remark B.1 with  $\underline{\Delta}/\lambda = 1/q$ , we transform the equation (83). Then, we get

$$\left( \frac{1}{f_1 g_1}, \frac{1}{f_1 \underline{g_1}} \right)_1 = \frac{\left( \frac{b_5}{f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}, \frac{b_8}{f_1} \right)_1}{\left( \frac{1}{b_1 f_1}, \frac{1}{b_2 f_1} \right)_1}, \quad (84)$$

$$\frac{(f_1 g_1, \overline{f_1 g_1})_1}{f_1 \overline{f_1}} = \frac{b_1 b_2 (b_5 g_1, b_6 g_1, b_7 g_1, b_8 g_1)_1}{q \left( \frac{g_1}{b_3}, \frac{g_1}{b_4} \right)_1}. \quad (85)$$

In the followings (B.2.1,  $\dots$ , B.2.4), we will consider the four types of deformation directions  $T_1, \dots, T_4$  in terms of variables  $g_1, \dots, g_4$ . We will present correspondence between these variables  $g_1, \dots, g_4$  by using Bäcklund transformations.

**B.2.1**  $T_1 : b_3 \mapsto b_3/q^2, \quad b_4 \mapsto b_4/q^2, \quad b_5 \mapsto qb_5, \quad b_6 \mapsto qb_6, \quad b_7 \mapsto qb_7, \quad b_8 \mapsto qb_8.$

This type corresponds to the direction considered in theorem B.4. And  $q$ -Painlevé equation is the equation (84). We will use this direction as the reference direction. We compare the other three types with this reference direction  $T_1$ .

**B.2.2**  $T_2 : b_1 \mapsto b_1/q, \quad b_8 \mapsto qb_8.$

This direction corresponds to  $s_2 T_1^{-1} s_2$ . By Padé method, the Lax pair for  $T_2$  are

$$L_2 : \left( b_8 x, \frac{x}{g_2} \right)_1 y(x) - (b_5 x, b_6 x)_1 y(qx) + c'_1 x (f_1 x)_1 \overline{y}(x) = 0, \quad (86)$$

$$L_3 : c'_2 x \left( \frac{\overline{f_1} x}{q} \right)_1 y(x) + \left( \frac{x}{b_1}, \frac{x}{q g_2} \right)_1 \overline{y}(x) - \left( \frac{b_7}{q} x, x \right)_1 \overline{y} \left( \frac{x}{a} \right) = 0 \quad (87)$$

where  $c'_1$  and  $c'_2$  are constant. And the  $q$ -Painlevé equation is given by

$$\left(\frac{1}{f_1 g_2}, \frac{1}{f_1 g_2}\right)_1 = \frac{\left(\frac{b_5}{f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}, \frac{1}{b_2 f_1}\right)_1}{\left(\frac{b_8}{f_1}, \frac{1}{b_1 f_1}\right)_1}. \quad (88)$$

Now, we derive the relation between  $g_1$  and  $g_2$ . We substitute  $x = 1/f_1$  for the equation (81) and the equation (86). And taking a ratio of these two equations, we get the following equation

$$\left(\frac{1}{f_1 g_1}, \frac{1}{f_1 g_2}\right)_1 = \frac{\left(\frac{b_5}{f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}\right)_1}{\left(\frac{1}{b_1 f_1}\right)_1}. \quad (89)$$

In terms of Bäcklund transformation, the relation (89) can be written as

$$g_2 = s_2 T_1^{-1}(g_1).$$

**Remark B.5** The direction  $T$  in section 3 is related with this direction  $T_2$  by exchange  $b_6$  and  $b_8$ . And similarly as above, it follows that

$$\left(\frac{1}{f_1 g_2}\right)_1 = \frac{\left(\frac{b_6}{f_1}\right)_1}{\left(\frac{b_8}{f_1}\right)_1} \left(\frac{1}{f_1 g}\right)_1. \quad (90)$$

**B.2.3**  $T_3 : b_1 \mapsto b_1/q, \quad b_3 \mapsto b_3/q, \quad b_4 \mapsto b_4/q, \quad b_5 \mapsto q b_5, \quad b_6 \mapsto q b_6, \quad b_7 \mapsto q b_7.$

This direction corresponds to  $T_2 = s_1 s_2 T_1 s_2 s_1$ . By Padé method, the Lax pair for  $T_3$  is

$$\begin{aligned} L_2 : (b_7 x)_1 y(x) - \left(\frac{x}{g_3}\right)_1 y(qx) + c'_1 x (f_1 x)_1 \bar{y}(x) &= 0, \\ L_3 : c'_2 x \left(\frac{b_5}{q} x, \frac{b_6}{q} x, \frac{\bar{f}_1 x}{q}\right)_1 y(x) + \left(b_5 x, b_6 x, \frac{b_7}{q} x, \frac{x}{b_1}\right)_1 \bar{y}(x) - \left(b_7 x, b_8 x, x, \frac{x}{g_3}\right)_1 \bar{y}\left(\frac{x}{q}\right) &= 0 \end{aligned}$$

where  $c'_1$  and  $c'_2$  are constant. And the  $q$ -Painlevé equation is given by

$$\left(\frac{1}{f_1 g_3}, \frac{1}{f_1 g_3}\right)_1 = \frac{\left(\frac{b_5}{f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}, \frac{1}{b_1 f_1}\right)_1}{\left(\frac{b_8}{f_1}, \frac{1}{b_2 f_1}\right)_1}. \quad (91)$$

Now, we derive the relation between  $g_1$  and  $g_3$ . Similarly as B.2.2, we get the following equation

$$\left(\frac{1}{f_1 g_3}\right)_1 = \frac{\left(\frac{1}{b_1 f_1}\right)_1}{\left(\frac{b_8}{f_1}\right)_1} \left(\frac{1}{f_1 g_1}\right)_1. \quad (92)$$

In terms of Bäcklund transformation, the relation (92) can be written as

$$g_3 = s_1 s_2(g_1).$$

**B.2.4**  $T_4 : b_3 \mapsto b_3/q, b_4 \mapsto b_4/q, b_5 \mapsto qb_5, b_8 \mapsto qb_8.$

This direction corresponds to  $T_4 = s_2 s_1 s_3 s_2 T_1^{-1} s_2 s_3 s_1 s_2$ . By Padé method, the Lax pair for  $T_4$  is

$$\begin{aligned} L_2 : \left( b_8 x, \frac{x}{g_4} \right)_1 y(x) - \left( b_6 x, \frac{x}{b_1} \right)_1 y(qx) + c'_1 x (f_1 x)_1 \bar{y}(x) &= 0, \\ L_3 : c'_2 x \left( \frac{\bar{f}_1 x}{q} \right)_1 y(x) + \left( b_5 x, \frac{x}{q g_4} \right)_1 \bar{y}(x) - \left( \frac{b_7}{q} x, x \right)_1 \bar{y} \left( \frac{x}{q} \right)_1 &= 0 \end{aligned}$$

where  $c'_1$  and  $c'_2$  are constant. And the  $q$ -Painlevé equation is given by

$$\left( \frac{1}{f_1 g_4}, \frac{1}{f_1 \underline{g_4}} \right)_1 = \frac{\left( \frac{1}{b_1 f_1}, \frac{b_6}{f_1}, \frac{b_7}{f_1}, \frac{1}{b_2 f_1} \right)_1}{\left( \frac{b_5}{f_1}, \frac{b_8}{f_1} \right)_1}. \quad (93)$$

Now, we derive the relation between  $g_1$  and  $g_4$ . Similarly as B.2.2, we get the following equation

$$\left( \frac{1}{f_1 g_4}, \frac{1}{f_1 g_1} \right)_1 = \left( \frac{b_6}{f_1}, \frac{b_7}{f_1} \right)_1. \quad (94)$$

In terms of Bäcklund transformation, the relation (94) can be written as

$$g_4 = s_2 s_1 s_3 s_2 T_1^{-1}(g_1).$$

## References

- [1] Hamamoto, T. and Kajiwara, K.: Hypergeometric solutions to the  $q$ -Painlevé equation of type  $A_4^{(1)}$ , J. Phys. A: Math. Theor. 40, 12509-12524 (2007)
- [2] Hamamoto, T., Kajiwara, K. and Witte, N.S.: Hypergeometric solutions to the  $q$ -Painlevé equation of type  $(A_1 + A'_1)^{(1)}$ , Int. Math. Res. Not. 2006 Article ID 84169 (2006)
- [3] Jimbo, M. and Sakai, H.: A  $q$ -analog of the sixth Painlevé equation. Lett. Math. Phys. 38, 145-154 (1996)
- [4] Kajiwara, K. and Kimura, K.: On a  $q$ -Difference Painlevé III Equation: I. Derivation Symmetry and Riccati Type Solutions, J. Nonlin. Math. Phys 10 86-102 (2003)
- [5] Kajiwara, K. and Kimura, K.: On a  $q$ -Difference Painlevé III Equation: II. Rational Solutions, J. Nonlin. Math. Phys. 10 282-303 (2003)
- [6] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y. and Yamada, Y.: Hypergeometric solutions to the  $q$ -Painlevé equations, Int. Math. Res. Not. 2497-2521 (2004)
- [7] Kajiwara, K., Noumi, M. and Yamada, Y.: A study on fourth  $q$ -Painlevé equation, J. Phys. A: Math. Gen. 34 8563-8581 (2001)
- [8] Masuda, T.: Hypergeometric  $\tau$ -functions of the  $q$ -Painlevé system of type  $E_7^{(1)}$ , SIGMA 5, 035, 30 pages (2009)

- [9] Masuda, T.: Hypergeometric  $\tau$ -functions of the  $q$ -Painlevé system of type  $E_8^{(1)}$ , Ramanujan J. 24 1-31 (2011)
- [10] Quispel, G. R. W., Roberts, J. A. G., Thompson, C. J.: Integrable mappings and soliton equations II, Physica D 34 183-192 (1989)
- [11] Ramani, A., Grammaticos, B. and Hietarinta, J.: Discrete Versions of the Painlevé Equations, Phys. Rev. Lett. 67, 1829-1832 (1991)
- [12] Ramani, A., Grammaticos, B. and Ohta, Y.: A Unified Description of the Asymmetric  $q$ - $P_V$  and  $d$ - $P_{IV}$  Equations and their Schlesinger Transformations, J. Nonlin. Math. Phys. 10(2), 215-228 (2003)
- [13] Sakai, H.: Casorati determinant solutions for the  $q$ -difference sixth Painlevé equation, Nonlinearity 11, 823-833 (1998)
- [14] Sakai, H.: Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Commun. Math. Phys. 220, 165-229 (2001)
- [15] Yamada, Y.: Padé method to Painlevé equations, Funkcial. Ekvac., 52, 83-92 (2009)
- [16] Yoshioka, R.: Padé approximation and special solution for  $q$ -Painlevé VI equation, Master thesis in Kobe University (Japanese) (2010)